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RECONSTRUCTION OF THE SURFACE TEMPERATURE OF ARCTIC GLACIERS FROM THE DATA OF TEMPERATURE MEASUREMENTS IN WELLS

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Consideration is given to the problem of reconstruction of the surface temperature of a glacier from the data of measuring the temperature in a well. Mathematically this problem is an inverse problem for a heat-conduction equation and refers to a number of incorrectly formulated (ill-posed) problems. For the reconstruction of the surface temperature, the Tikhonov regularization method has been used. A model that takes into account the vertical advection of the annual layers has been adopted as a mathematical model that describes the propagation of heat in a glacier. The boundary temperatures have been reconstructed from the results of temperature measurements in wells obtained for certain glaciers of the Arctic. The effect of the initial temperature distribution, the accumulation rate, and the magnitude of the geothermal heat flux on the reconstructed boundary temperature has been investigated.

Introduction. The deviation of temperature in a glacier from the stationary distribution is caused by climatic changes. In particular, glaciers contain information on changes in the temperature on the glacier surface that occurred in the past. Arctic changes in the ambient temperature have been investigated for the past 100–150 years. In the period from 1840 to the mid-20th century, a warming of the climate that varied from 1 to 3^{o}C at different places of the region and 1.5^{o}C averaged 1.5^{o}C was used [1]. The measured temperature profiles in a well indicate more significant changes in the surface temperature of glaciers.

One traditional method of reconstruction of the ambient temperature in the past from the data of well measurements is based on the determination of the relative concentration of stable isotopes in the corresponding time layer of ice, for example, of the relative concentration of the oxygen isotope δ^{18}O. Oxygen isotopes are accumulated in the corresponding annual (time) layer of ice at the moment of its formation from precipitation, falling out onto the glacier surface at the corresponding instant of time. The relative concentration δ^{18}O in the atmosphere linearly depends on the air temperature [2]. Two coefficients of this dependence can be obtained using the method of "volumetric gradient" or using simultaneous measurements of the atmospheric temperature T and the relative concentration δ^{18}O [13]. Cuffey et al. [4, 5] calibrated an isotopic paleothermometer (i.e., established the coefficients of the linear dependence $T(\delta^{18}\text{O})$) for the central part of Greenland. This enabled them to determine the atmospheric temperature in the past 500–600 years from the data of measurements of the relative concentration of δ^{18}O. It was noted [6] that the results of the reconstruction of the climate in the past obtained by the method that is based on measuring the relative concentration δ^{18}O generally correspond to the salient features of the temperature change in a well, but in some cases difficulties with the interpretation of the data of the δ^{18}O measurements arise.

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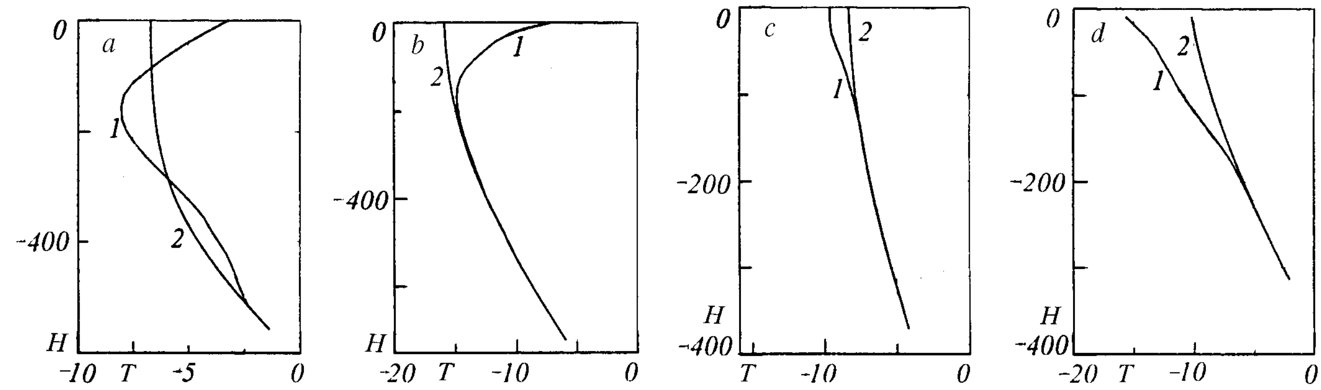


Fig. 1. Temperature profiles in wells of the Austfonna (a), Akademiya Nauk (b), Barnes Icecap (c), and Gulia (d) Glaciers: 1) temperature profile obtained as a result of its measurements in the well; 2) stationary temperature distribution in the glacier.

Another method of reconstruction of the temperature of the glacier surface is based on control methods [7]. This is one method of solution of inverse problems, which is as follows in this case. It is assumed that the temperature profile in a well $T(z, t_f)$ is determined by the change in the surface temperature $\mu(t)$ and by other parameters of the glacier according to the well-known approximation that describes the process of propagation of heat in a glacier (here z is the vertical coordinate, t is the time, and t_f is the final instant of time corresponding to temperature measurements in the well). A mathematical dependence of the temperature profile $T(z, t_f)$ on the surface temperature can be expressed using the operator equation $T(z, t_f) = R\{\mu\}$. It is also assumed that there exists an inverse operator, i.e., the surface temperature can be established from the equation $\mu(t) = R^{-1}\{T(z, t_f)\}$. The control method makes it possible to determine the optimum solution that corresponds to a minimum deviation from the unknown exact solution [7]. In this investigation, the glacier height did not change with time. The effect of a change in the glacier height on the reconstructed surface temperature has been discussed in [8, 9]. We note that the stability of the solution of the inverse problem obtained by the control method is not proved. But it is known that the mapping of a certain set $F(\mu(t) \in F)$ onto the set $G(T(z, t_f) \in G)$ realized by the operator R is not bicontinuous, i.e., the solution of the inverse problem is unstable to small disturbances perturbations of the temperature profile $T(z, t_f)$ and the inverse problem is an incorrectly formulated (ill-posed) problem.

To reconstruct the surface temperature from the data of well measurements performed for certain glaciers of the Arctics, we used the Tikhonov regularization method, which makes it possible to find a solution of the inverse problem that is stable to slight changes in the input data [10].

In the problem of reconstruction of the surface temperature, we used the following data as the input data: the temperature profile measured in the well, the coefficients of thermal conductivity and thermal diffusivity, the geothermal heat flux at the base of the glacier, the rate of accumulation of precipitation, and the vertical velocity of motion of annual layers in the glacier. Using mathematical modeling, we can establish correlations between the vertical velocity of motion of the annual layers and the reconstructed surface temperature.

The surface temperature implies the temperature at a depth of 10 m [11]. The annual fluctuations of the atmospheric temperature can penetrate to only a limited depth that depends on the site of location of the investigated region of the glacier and on time. As a rule, this depth is about 10 m.

Experimental Data. In the present work, we have used temperature profiles that were measured in wells drilled in certain glaciers in the Arctics (Fig. 1) in the period from 1975 to 1992. The accuracy of measuring the temperature in a well is about 0.01°C. The range in which the rate of accumulation of precipitation can be found is from 0.1 to 0.6 m/year. It has also been assumed that advection changes linearly as a function of the value of the accumulation rate on the surface to zero at the base of a glacier. This approximation corresponds to the model of a glacier with uniform properties, i.e., with constant density and coefficients of heat capacity and thermal conductivity [5]. The best results of the reconstruction of the surface temperature were obtained in those cases where the temperature measured in the lower part of the glaciers near the bases corresponds to the stationary distribution. The geothermal heat flux for these glaciers can be determined directly using the measured temperature profile, and we assume that, in the past, the geothermal heat flux had the same value as during the well measurements.

Regularization Method for the Problem of Reconstruction of the Glacier-Surface Temperature. Before proceeding to consideration of the method of solution of the inverse problem, we formulate the primal problem. Its mathematical formulation involves a one-dimensional heat-conduction equation and initial and boundary conditions. The main factors that have an effect on the temperature distribution in a glacier have been considered in [12, 13]. In this work, use was made of the mathematical formulation of the problem [7]

$$\begin{gathered} T_t + wT_z = \chi T_{zz}\,, \quad 0 < t < t_\mathrm{f}\,, \quad 0 < z < H\,, \\ T(z, 0) = T_0(z)\,, \quad 0 < z < H\,, \\ T(0, t) = \mu(t)\,, \quad 0 < t \le t_\mathrm{f}\,, \\ -kT_z(H, t) = Q(t)\,, \quad 0 < t \le t_\mathrm{f}\,, \end{gathered} \tag{1}$$

where $T_0(z)$ is the temperature profile at the initial instant of time in the past; the subscripts z and t denote differentiation with respect to the time and space coordinates, respectively. In the heat-conduction equation, we disregard the terms whose presence is attributed to the work of the force of gravity, which causes the vertical advection of ice layers and their spreading in the horizontal direction under the pressure of the overlying layers [5]. We consider glaciers with a depth of the order of 500 m, whereas the terms indicated above become significant at a depth larger than 800 m [14].

As has already been noted, the solution of the primal problem can be written in the form of the operational relation $\theta(z) = R\{\mu\}$. Then the solution of the inverse problem can formally be written in the form $\mu(t) = R^{-1}\{\theta(z)\}$. If $\theta(z)$ is the temperature profile obtained as a result of measurements, it contains temperature disturbances that result in $\theta(z) \notin G$, where $G = RF$ is the set of transforms in the mapping realized by the operator R, and $\mu \in F$. These disturbances are due to measurement errors, and also to the fact that the considered mathematical model does not take into account all possible processes that have an effect on the temperature distribution in the glacier. Hence, the equation $\mu(t) = R^{-1}\{\theta(z)\}$ has no exact solution for the element $\theta(z)$.

To eliminate this difficulty, we introduce the notion of the quasi-solution $\tilde{\mu}(t) \in F$ as a function that minimizes the integral

$$\int_0^H (R\{\tilde{\mu}(t)\} - \theta(z))^2\, dz = \min \left[\int_0^H (R\{\mu(t)\} - \theta(z))^2\, dz \right] \equiv \alpha\,. \tag{2}$$

In this case, the set F represents a set of continuous functions with assigned values at the ends of the segment $[0, t_\mathrm{f}]$. The quasi-solution $\tilde{\mu}(t)$ is the boundary temperature for which the solution of problem (1) has a minimum deviation from the measured profile $\theta(z)$ in the metric L_2.

The problem of minimizing the discrepancy (selecting a quasi-solution) in the general case does not allow the determination of the quasi-solution $\tilde{\mu}(t)$ in a stable manner with respect to the small disturbances of the profile $\theta(z)$. The sufficient conditions of the correctness of the problem of minimizing the discrepancy α are the following conditions: 1) the set F must be compact and convex; 2) any sphere in the space U, $\theta(z) \in U$ is strictly convex [10]. Condition 2) can be considered to be fulfilled since the experimental profiles belong to the set of continuous functions $C[0, H]$. Thus, the problem of selecting a quasi-solution becomes correct on any compact subset of the set of continuous functions with assigned values at the ends of the segment $[0, t_f]$. We note that the inverse problem can also be considered on compact sets of functions that are not continuous on the segment $[0, t_f]$ [6].

Apart from the selection method, there are regularization methods that allow a stable determination of the solution for the inverse problem [10]. In this work, we used the Tikhonov regularization method, which involves the determination of the function $\mu(t)$ that minimizes the following functional:

$$\Psi = \int_0^H (R\{\mu(t)\} - \theta(z))^2\, dz + \beta\cdot\Omega[\mu(t)]\,, \tag{3}$$

where β is the regularization parameter coordinated with the accuracy of the input data. The functional $\Omega(\mu(t))$ is referred to as stabilizing or as a stabilizer:

$$\Omega(\mu) = \int_0^{t_f} \sum_{j=0}^{m_0} q_j(t) \left(\frac{d^j\mu}{dt^j}\right)^2 dt\,, \tag{4}$$

where the coefficients $q_j \geq 0$ and $q_{m_0} > 0$. In this work, use was made of the stabilizer of first or second order ($m_0 = 1$ or 2).

Algorithm of Solution of the Inverse Problem. The procedure of minimizing the functional Ψ was realized using the gradient method and represents an iteration procedure. At the first step of the iterations, we assign a zero approximation of the boundary temperature in the form of a grid function for a certain subdivision of the segment $[0, t_f]$: $(\mu(t_0), \mu(t_1), ..., \mu(t_k),)^0 \equiv \vec{\mu}^0$. At the nth step of iteration, the values of the boundary temperature are determined from the relation

$$\vec{\mu}^{n+1} = \vec{\mu}^n - \gamma^n\cdot\mathrm{grad}\,\Psi(\vec{\mu}^n)\,,$$

where $\gamma^n > 0$ is the gradient step. The zero approximation of the boundary temperature represents a linear function that assumes assigned values at the ends of the segment $[0, t_f]$: $\mu(0)$ and $\mu(t_f)$. As an initial temperature distribution, use was made of the stationary profile. Accordingly, $\mu(0)$ represents the value of the surface temperature in the case of the stationary distribution of the temperature in a glacier. To determine the profile at the nth step of iteration $R\{\vec{\mu}^n\} = T^n(z, t_f)$, we used an implicit-difference scheme. This profile is necessary for calculating the value of the functional $\Psi^n = \int_0^H (R\{\vec{\mu}^n\} - \theta(z))^2\, dz + \beta\cdot\Omega[\vec{\mu}^n]$, which is compared to the previous value Ψ^{n-1} in order to control the fulfillment of the condition $\Psi^n < \Psi^{n-1}$. The derivatives $\partial\Psi^n(\mu_j^n)/(\partial\mu_j^n)$ can be determined if the values of the derivatives $\partial R\{\mu^n\}/\partial\mu_j^n = \partial T^n/\partial\mu_j^n$ are known. The latter derivatives are the profiles $W_j = \partial T^n/\partial\mu_j^n$ that are the solutions of the problem for the heat-conduction equation obtained from problem (1) by differentiating all relations (1) with respect to μ_j^n, i.e., the profiles W_j are the solutions of problem for the heat-conduction equation (1) with trespect to he zero boundary condition at the base of the glacier $\frac{\partial W_j}{\partial z}\Big|_{z=H} = 0$, the zero initial condition $W_j(z, 0) = 0$, and a surface temperature that is equal to zero at all instants of time τ_i except for the instant τ_j, when the boundary temperature equals 1.

The iteration procedure is performed until we reach the minimum of the functional Ψ (with assigned accuracy) to which the optimum quasi-solution of the inverse problem corresponds.

Investigation of the Factors Affecting the Solution of the Inverse Problem. Preparatory to reconstructing the surface temperature, we investigate the solution of the primal problem in order to answer the following questions:

1) What is the effect of the initial (unknown) temperature distribution in the glacier on the reconstructed surface temperature in the case where we know just the scale of the temperatures in the past that,, as we assume, is a magnitude of the same order as at present (for the investigated time intervals)?

2) What is the structure of the temperature distribution in the glacier for certain boundary temperature regimes in the capacity of which we will consider harmonically oscillating boundary temperatures?

We note that there is a hypothesis [7] on climatic changes in the past according to which these changes were of a periodic nature. Furthermore, we will take into consideration the possibility of representating the boundary temperature in the form of a Fourier expansion. From the structure of the profile that corresponds to a harmonically oscillating temperature, one can draw a conclusion on the efficiency of the reconstruction of the surface temperature as a function of the periodicity of its oscillations.

The solution of problem (1) can be represented in the form of a superposition of three profiles: $T(z, t) = T_1(z, t) + T_2(z, t) + T_3(z, t)$, where each of the functions on the right-hand side of the equation represents, respectively, the solution of one of the following problems:

$$\text{(a)}\ \frac{\partial T_1}{\partial t} = \chi \frac{\partial^2 T_1}{\partial z^2} - w(z)\frac{\partial T_1}{\partial z}, \quad T_1(0, t) = 0, \quad \frac{\partial T_1}{\partial z}(H, t) = 0, \quad T_1(z, 0) = T_0(z);$$

$$\text{(b)}\ \frac{\partial T_2}{\partial t} = \chi \frac{\partial^2 T_2}{\partial z^2} - w(z)\frac{\partial T_2}{\partial z}, \quad T_2(0, t) \equiv 0, \quad \frac{\partial T_2}{\partial z}(H, t) = -\frac{Q}{k}, \quad T_2(z, 0) = 0;$$

$$\text{(c)}\ \frac{\partial T_3}{\partial t} = \chi \frac{\partial^2 T_3}{\partial z^2} - w(z)\frac{\partial T_3}{\partial z}, \quad T_3(0, t) = \mu(t), \quad \frac{\partial T_3}{\partial z}(H, t) = 0, \quad T_3(z, 0) = 0.$$

We investigate the analytical solutions of these problems for $w(z) = 0$ and after that sum up the obtained results for the case where $w(z) \neq 0$.

The solution of problem (a) makes it possible to determine the effect of the unknown initial temperature distribution $T_0(z)$ on the final profile $T(z, t_f)$ and also the time $t_0 \in (0, t)$ after which this effect can be disregarded. This solution has the form [15]

$$T_1(z, t) = \frac{2}{H} \sum_{n=0}^{\infty} \exp\left(-\frac{\chi (2n+1)^2 \pi^2 t}{4H^2}\right) \cos\frac{(2n+1)\pi z}{2H} \int_0^H T_0(x) \times$$

$$\times \cos\frac{(2n+1)\pi x}{2H}\, dx \quad (0 < z < H,\ t > 0), \tag{5}$$

where x is the integration variable. If $t \to \infty$, the solution $T_1(z, t) \to 0$. Hence, t_0 can be determined from the inequality $\max\limits_{0<z<H,\, t>t_0} |T_1(z, t)| < \varepsilon$. Let ε satisfy the inequality $|\theta(z) - T(z, t_f)| < \varepsilon$, where $\theta(z)$ is the measured temperature profile, and the profile $T(z, t_f)$ corresponds to Ω — the optimum quasi-solution $\tilde{\mu}(t)$ (ε is determined by measurement errors and the adequacy of the mathematical model to the actual physical processes in the glacier). Then the effect of the initial temperature on the final distribution $T(z, t_f)$ and on the reconstructed boundary temperature $\mu(t)$ can be disregarded when $t > t_0$ ($t_0 \le t \le t_f$). For the solution (5) we have the following estimate:

$$\max_{0<z<H, t>t_0} |T_1(z, t)| \le 2M \sum_{n=0}^{\infty} \exp\left(-\frac{\chi (2n+1)^2 \pi^2 t_0}{4H^2}\right) = 2M \cdot S, \quad \text{where} \quad M = \max_{0<z<H} |T_0(z)| \} \ . \tag{6}$$

The quantity

$$\delta = \frac{\max\limits_{0<z<H, t>t_0} |T_1(z, t)|}{\max\limits_{0<z<H} |T_0(z)|} \le 2 \exp\left(-\frac{\chi\pi^2 t_0}{4H^2}\right) + \frac{H}{\sqrt{\chi\pi^2 t_0}} \operatorname{erfc}\left(\frac{\pi\sqrt{\chi t_0}}{2H}\right),$$

where $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$, $\operatorname{erf}(x)$ being the integral of errors, determines the degree of influence of the initial temperature distribution on the solution $T_1(z, t)$. The values of the quantity δ for certain instants of time t_0 equal: 1) $\delta \approx 2 \cdot 10^{-1}$, $t_0 = 5 \cdot 10^3$ years; 2) $\delta \approx 2.6 \cdot 10^{-2}$, $t_0 = 10^4$ years; 3) $\delta \approx 3.3 \cdot 10^{-4}$, $t_0 = 2 \cdot 10^4$ years (for $H = 500$ m).

The presence of advection in the glacier does not increase the value of the instant of time t_0 that corresponds to the selected δ. Indeed, the presence of the term $-w(z)\frac{\partial T}{\partial z}$ in Eq. (1) can be interpreted as a consequence of the nonuniformity of the medium's thermal properties, setting $-w(z) = \partial\chi/\partial z < 0$ ($\chi = \chi(z)$). Near the base $z \approx H$ the velocity of advection equals zero ($w(z) \approx 0$) and the process of propagation of heat is described by the heat-conduction equation without advection and with the value of the thermal-diffusivity coefficient $\chi(H) \approx \chi_{gl}$. The values of the thermal-diffusivity coefficient $\chi(z)$ decrease from surface to base by virtue of the inequality $\partial\chi/\partial z < 0$ to the value $\chi(H) = \chi_{gl}$, i.e., on average $\bar{\chi}(z) > \chi_{gl}$. Thus, we have the following estimate $\max\limits_{0<z<H} |\bar{T}_1(z, t)| \le \max\limits_{0<z<H} |T_1(z, t)| \}$, since the sum of the series (6) decreases as χ increases. In this inequality, $\bar{T}_1(z, t)$ represents the solution of problem (a) with account taken of advection. Quantitative estimates with account for advection were obtained in [16]. For example, for the linear advection profile and a glacier height of $H = 1000$ m, a noticeable difference in the temperature profiles obtained for zero and linear advection respectively is observed several decades later. After 100 years the maximum deviation amounts to ~10%, and after 300 years the deviation amounts to ~30%. These results are obtained for the value of the velocity of accumulation rate of 0.3 m/year.

The solution of problem (b) is known [15]:

$$T_2(z, t) = \frac{Qz}{k} - \frac{8QH}{k\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \exp\left(-\frac{\chi (2n+1)^2 \pi^2 t}{4H^2}\right) \sin\frac{(2n+1)\pi z}{2H} \ . \tag{7}$$

For the nonstationary side of the solution (7) for $t \geq t_0 = 2{\cdot}10^4$ years we have the following estimate: $\max\limits_{0<z<H,\, t\approx t_0} \left| T_{2\mathrm{nonst}}(z, t) \right| \approx \dfrac{8QH}{k\pi^2} \exp\left(-\dfrac{\chi\pi^2 t_0}{4H^2}\right) \approx 2.2{\cdot}10^{-3}$. Thus, on condition that $t_0 \leq t < t_\mathrm{f}$, a stationary solution which represents a linear distribution of the temperature Qz/k can be taken as the solution of problem (b). With account taken of advection, the stationary solution of problem (b) has the form

$$T_{2\mathrm{st}}(z) = -\frac{Q \int\limits_0^z dx' \exp\left(\int\limits_0^{x'} \tilde{w}(x'')\, dx''\right)}{k \exp\left(\int\limits_0^H \tilde{w}(x')\, dx'\right)}, \quad \tilde{w}(z) = \frac{w(z)}{\chi}. \tag{8}$$

The solution (8) was used as the initial distribution in reconstruction of the boundary temperatures of the investigated glaciers.

Next we consider problem (c). We represent the boundary temperature in the form of a Fourier expansion on the segment $0 \leq t \leq t_\mathrm{f}$:

$$\mu(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{t_\mathrm{f}}, \quad b_n = \frac{2}{t_\mathrm{f}} \int\limits_0^{t_\mathrm{f}} \mu(t) \sin\frac{n\pi t}{t_\mathrm{f}}\, dt. \tag{9}$$

Then for $t \geq t_0$ the solution of problem (c) can be represented in the form of a superposition of steady-state solutions corresponding to individual harmonics [15]:

$$T_3(z, t) = \sum_{n=1}^{\infty} A_n(z)\, b_n \sin(\omega_n t + \varphi_n(z)), \tag{10}$$

where

$$A_n(z) = \sqrt{\frac{\cosh 2p_n(z/H - 1) + \cos 2p_n(z/H - 1)}{\cosh 2p_n + \cos 2p_n}},$$

$$p_n = \sqrt{\frac{\omega_n}{2\chi}}\, H, \quad \omega_n = \frac{\pi n}{t_\mathrm{f}}, \quad \varphi_n(z) = \arg\frac{\cosh(p_n(z/H - 1)(1 + i))}{\cosh(p_n(1 + i))}.$$

The amplitudes $A_n(z)$ of the oscillations penetrating into the glacier decrease as the coordinate z increases according to a nearly exponential law, so much the faster the higher the frequency of oscillations of the boundary temperature. When $n \geq 4{\cdot}10^4$ ($t_\mathrm{f} = 2{\cdot}10^4$), the oscillations of the boundary temperature that represent seasonal oscillationsdo not penetrate into the glacier to a depth larger than ~(0.01–0.03)$\cdot H$ for various thermal properties of the firn. This depth is approximately 15 m. It should also be taken into account that the value of the thermal-diffusivity coefficient χ for compacted snow (firn) is smaller than the corresponding value for ice, and the amplitudes $A_n(z)$ for smaller values of the coefficient χ decrease faster with increase in the coordinate. Thus, we can exclude these oscillations from consideration, setting $b_n = 0$ for $n > n_0 = 4{\cdot}10^4$ in (9), i.e., we represent the surface temperature in the form of a trigonometric polynomial of the n_0th degree.

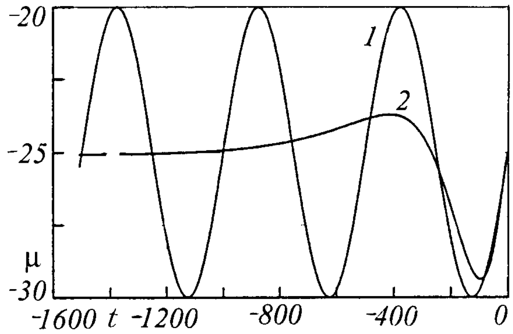


Fig. 2. Initial surface temperature, which oscillates harmonically (1), and the boundary reconstructed by the regularization method from the profile that corresponds to the initial boundary temperature (2). μ, ^{o}C; t, years.

For rather large n, the phase of steady-state oscillations of the temperature has a linear dependence on the coordinate except for a small region near the base of the glacier ($0.9 < z/H < 1$), so that $\varphi(z) \approx -az$. In this case, the temperature oscillations in the glacier that are formed as a result of harmonic oscillations of the surface temperature represent a temperature wave with an exponentially decaying amplitude:

$$T_3(z, t) \xrightarrow[n \to \infty]{} \exp\left(-\sqrt{\frac{\omega_n}{2\chi}}\, z\right) \sin\left(\omega_n t - \sqrt{\frac{\omega_n}{2\chi}}\, z\right), \quad a \xrightarrow[n \to \infty]{} \sqrt{\frac{\omega_n}{2\chi}}\,. \tag{11}$$

The approximation of the temperature wave (11) can actually be used on condition that $p_n = \sqrt{\omega_n/2\chi}\,H \geq 2\pi$. Then the phase velocity and the wavelength for $p_n \geq 2\pi$ can be obtained from expression (11): $V_{\mathrm{ph}n} = \sqrt{2\chi\omega_n}$, and $\lambda_n = 2\pi\sqrt{2\chi/\omega_n}$. Taking into account that $p_n = \sqrt{\omega_n/2\chi}\,H$, we obtain $\lambda_n/H = 2\pi/p_n$. Therefore, the condition $p \geq 2\pi$ (or $T_\omega \leq H^2/(4\pi\chi)$, where T_ω is the period of oscillations of the boundary temperature), is equivalent to the condition $\lambda \leq H$, which means that the profile contains information on at least one period of oscillations of the boundary temperature.

If the length of the temperature wave representing the profile for the harmonically oscillating temperature of the glacier surface $\mu(t) = A \sin \omega t$ is commensurable to the height of the glacier $H(\lambda \approx H)$, it is obviously impossible to reconstruct more than one period of oscillations since the profile contains information on just one period of these oscillations.

Let $\lambda < H$. In this case, taking into account the exponential decay of the amplitude of the temperature wave $A(\xi) = A \exp(-p\xi)$, $\xi = z/H$, at the distance λ from the surface of the glacier for the amplitude of oscillations of the boundary temperature $A \approx 5^{o}$C (which corresponds to the actual changes in the surface temperature of the considered glaciers) the amplitude of the temperature wave turns out to be $A(\lambda) = A \exp(-2\pi) \approx 0.01^{o}$C. This value is, first, comparable to the error of temperature measurements in the well, and, hence, any fluctuations of the temperature profile with the amplitude $A \approx 0.01^{o}$C must be considered as a measurement error. Second, the sensitivity of this method does not allow the reconstruction of the boundary temperature from the fluctuations of the profile with amplitudes of the order of 0.01^{o}C or smaller. Hence, an the condition that $\lambda < H$ it is impossible to reconstruct more that one period of oscillations of the boundary temperature, too.

One can draw a final conclusion that on condition that $T_\omega \leq H^2/(4\pi\chi)$, where T_ω is the period of oscillations of the initial boundary temperature, one period of these oscillations will be reconstructed (qualitatively).

On condition that $\lambda > H$, the boundary temperature should be considered not as oscillating but as monotonously changing. In this case, the result of reconstruction reflects the tendency toward a change in the

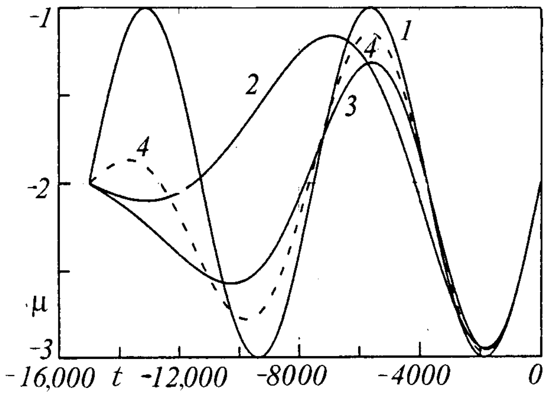


Fig. 3. Initial boundary surface temperature, which oscillates harmonically (1), and the boundary temperatures reconstructed using a trigonometric polynomial for the following values of the order of a stabilizer, respectively: 2) $m_0 = 0$; 3) 3; 4) 5.

initial boundary temperature. At the same time, just as in the previous cases, it is obviously impossible to reconstruct more than one period of oscillations of the initial boundary temperature.

In the case of a rapidly oscillating surface temperature ($\lambda < H$), the efficiency of its reconstruction turns out to be rather low. In fact, using the algorithm described above, we are able to reconstruct only the second half of the last period, and the first half of this period can be reconstructed just qualitatively (Fig. 2).

The efficiency of the reconstruction of the boundary temperature in the case of its changes being of an oscillating nature can be improved if a trigonometric polynomial is selected as a method of approximation of the sought surface temperature, i.e., it is possible to determine not the values of the surface temperature at the subdivision nodes but the coefficients of the trigonometric polynomial approximating the sought boundary temperature. In this case, without account for advection the solution (10) can be taken as a solution of the primal problem $T(z, t_f) = R\{\mu\}$. Then the functional Ψ can be considered as a function of the coefficients b_n of the trigonometric polynomial. We write the condition of the minimum of the function $\Psi(b_n)$ in the form $\partial\Psi/\partial b_n = 0$, $n = 1, ..., n_0$. After calculating the derivatives, we obtain a system of linear algebraic equations for b_n.

Using this method of approximation, we have reconstructed the boundary temperature regime: $\mu(t) = \sin \frac{4\pi t}{t_f}$, where $t_f = 1.5 \cdot 10^4$ years (Fig. 3). We note that for this periodicity of oscillations the wavelength is $\lambda > H$ ($H = 500$ m), i.e., the temperature profile does not contain information on more than one period of oscillations. Moreover, in this case the profile actually contains information on just the last increase in the surface temperature, which corresponds to the last quarter of the period of oscillations. The coincidence of the initial and reconstructed boundary temperatures in the region of $t < -2000$ years is attributed to the appropriate selection of the method of approximation. The success of reconstruction of the given temperature regime depends on the order of the stabilizer m_0. The quality of the reconstructed surface temperature changes with increase in the order of the stabilizer. For the given periodicity of oscillations, the best coincidence between the initial and reconstructed boundary temperatures is observed for $m_0 = 5$.

Reconstruction of the Surface Temperature from Experimental Data. Boundary temperatures reconstructed for the Austfonna, Akademiya Nauk, Barnes Icecap, and Gulia Glaciers are presented in Fig. 4. Changes in the reconstructed temperatures are of similar nature in the corresponding pairs. The reasons for the asynchronous response to the global climatic changes in the northern and southern hemispheres were discussed earlier [17]. We also note that temperature changes at a depth of 10 m in these glaciers have been

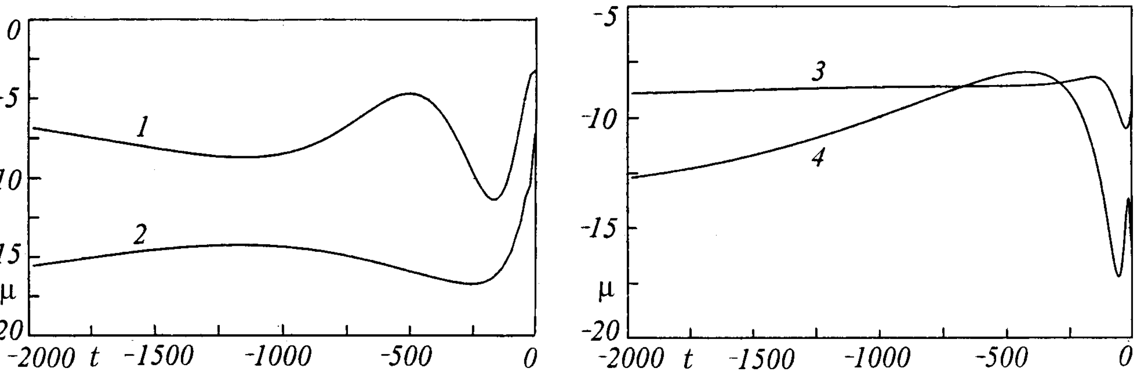
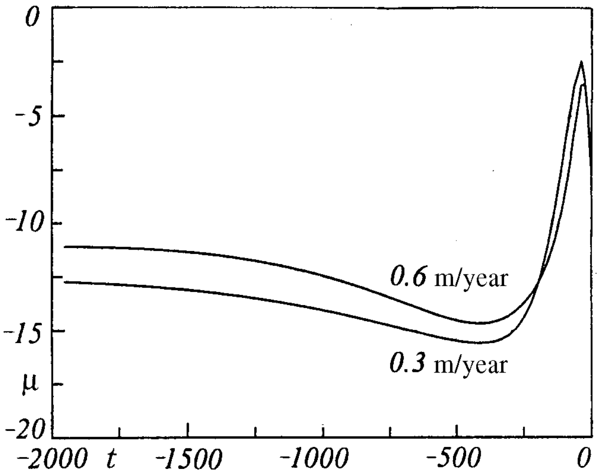


Fig. 4. Boundary temperatures reconstructed for, respectively, the Austfonna (1), Akademiya Nauk (2), Barnes Icecap (3), and Gulia (4) Glaciers.



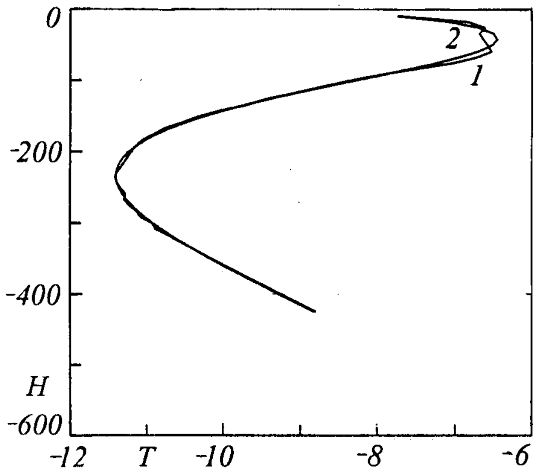


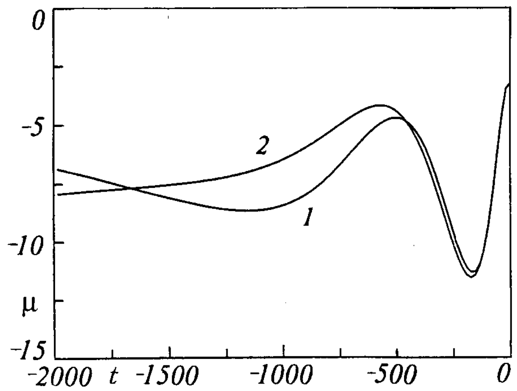
Fig. 5. Boundary temperatures reconstructed for different values of the accumulation rate (the Franz Josef Land).

Fig. 6. Temperature profile obtained as a result of its measurements in a well on the Franz Josef Land (1) and the profile corresponding to the reconstructed surface temperature (2).

much more significant in the past 200 years than the corresponding changes in the air temperature determined using an isotopic analysis [1]. This is, apparently, due to the complex interaction between the water formed as a result of melting of ice or owing to the fallout of wet precipitation onto the glacier surface and firn. Thus, the temperature at a depth of 10 m can change more significantly than the air temperature.

The method proposed was also used for reconstructing the surface temperature based on the data (profiles) obtained fairly recently on the Franz Josef Land Glacier. The value of the rate of accumulation of precipitation (0.3–0.6 m/year) has actually not affected the reconstructed surface temperature in the past 200 years, since this period of time corresponds to the data in the upper part of the measured temperature profile where the total (integral) effect of advection is insignificant (Fig. 5). The deviation of the profile that corresponds to the reconstructed surface temperature from an experimental profile does not exceed 5%, except for a small region near the surface, which is, probably, due to measurement errors (Fig. 6). The last period in which cooling occurred is similar to the corresponding periods for the Barnes Icecap and Gulia Glaciers.

The temperature profile in a well at the initial instant of time $t = 0$ is one parameter of the problem. It was noted above that when $t_0 < t < t_f$ the solution of the inverse problem is insensitive to these initial data.



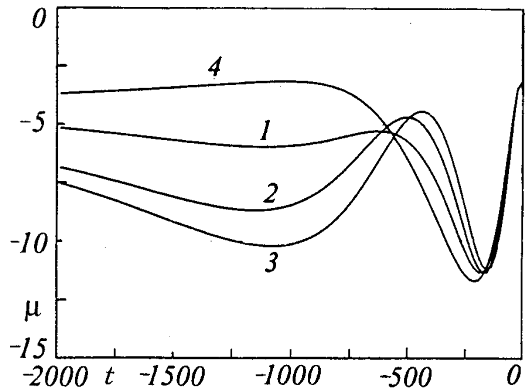


Fig. 7. Boundary temperatures reconstructed for different values of the initial temperature on the surface of the Austfonna Glacier: 1) $\mu(0) = -7.5^{o}C$; 2) -30.

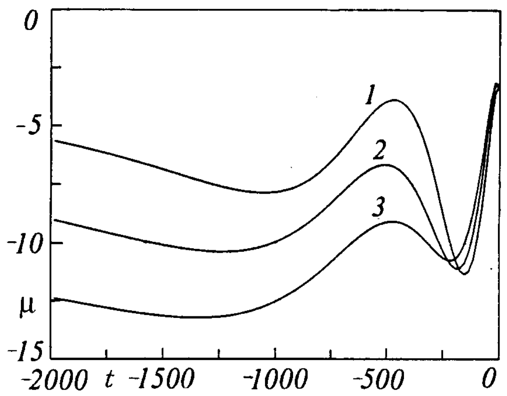
Fig. 8. Boundary temperatures reconstructed for different values of the geothermal flux: 1) $Q = 0.035$; 2) 0.05; 3) 0.06 W/m^2; 4) a constant temperature equal to the phase-transition temperature is assigned at the glacier–rock interface.

For the Austfonna Glacier, the boundary temperature was reconstructed for different initial conditions; we took, as such conditions, the stationary profiles with a surface temperature equal to, respectively, -7.5 and -30^{o}C. This difference in the initial conditions has not substantially affected the reconstructed surface temperature in the past 500–600 years (Fig. 7).

In the lower part, near the base, all the glaciers considered, except for the Austfonna Glacier, affected are in the stationary state. The values of the geothermal heat fluxes were obtained from the corresponding temperature distributions near the glaciers' bases. In the case of the Austfonna Glacier, it is impossible to determine the geothermal heat flux by the method proposed, since the entire glacier is in the nonstationary state. We assumed that the value of the geothermal heat flux for the Austfonna Glacier was in the interval $0.03 \le Q \le 0.06$ W/m^2 and reconstructed the surface temperature for some values from this interval (0.035, 0.05, and 0.06 W/m^2). Noticeable differences in the reconstructed temperatures are observed for the instants of time $t < -200$ years (Fig. 8).

Once the well was drilled on the Austfonna Glacier the level of water in the well rose by 50 m. This enables us to assume that the temperature at the glacier–rock interface reaches the melting point of ice, which can be due to the existence of the flows of sea water along the interface. The low melting temperature of ice (-1.4^{o}C) is attributed to the presence of the impurities of salts in this water. The reconstructed surface temperature for the case where, at the interface, a constant temperature (-1.4^{o}C) is assigned instead of the heat flux is presented in Fig. 8 (curve 4). This boundary temperature corresponds to the surface temperature reconstructed for a geothermal heat flux of 0.029 W/m^2. Assuming that the geothermal heat flux from the rock is known, we can evaluate the ablation rate at the interface. The difference of the values of the heat fluxes at the interface is expended on melting ice and is equal to $\rho L\dot{a}$, where ρ is the density of the ice, L is the latent heat of melting, and $\dot{a}$ is the melting rate of the ice. If the geothermal heat flux from the rock is equal to 0.05 W/m^2, then $\dot{a} = 2.1$ mm/year and $\dot{a} = 7.3$ mm/year for the largest possible value of the geothermal heat flux equal to 0.1 W/m^2.

The effect of the accumulation rate (with account for the linear advection profiles) on the reconstructed surface temperature for the Austfonna Glacier is similar to the corresponding effect of accumulation for the Franz Josef Land Glacier: there have been no large differences in the reconstructed surface temperatures in the past 200 years (Fig. 9).



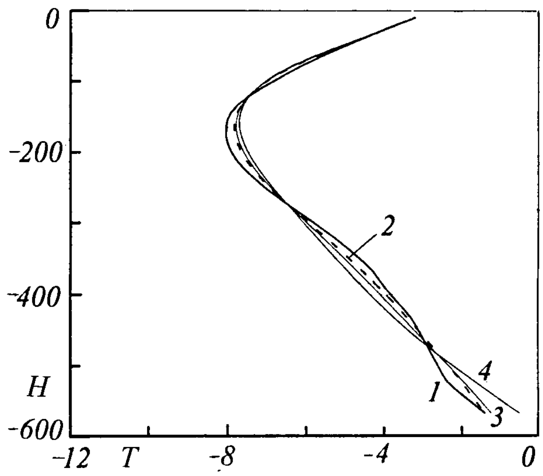


Fig. 9. Boundary temperatures reconstructed for different values of the accumulation rate (the Austfonna Glacier): 1) 0.1 m/year; 2) 0.3; 3) 0.7.

Fig. 10. Temperature profiles in the Austfonna Glacier: 1) experimental values; 2, 3, and 4) profiles corresponding to the reconstructed boundary temperatures (0.1, 0.3, and 0.7 m/year, respectively).

The behavior of the temperature at the base of the Austfonna Glacier is of a rather complicated nature (Fig. 10, curve 1). In this region we can, apparently, observe a certain interaction of the temperature regimes. As was noted above, melting is likely to occur at the interface (the dashed curve in Fig. 10). In this case, in order to model the propagation of heat, we used different boundary conditions for $z = H$ (Fig. 10). Curves 3 and 4 correspond to $Q = 0.035$ and 0.06 W/m^2.

CONCLUSIONS

1. The regularization method for reconstructing the temperature at a depth of 10 m can be considered as alternative relative to the control method and the method that is based on an isotopic analysis. It makes it possible to stably determine the solution of the inverse problem relative to small disturbances of the input data (the temperature profile).

2. The temperature in a glacier at a depth of 10 m is very sensitive to climatic changes. This is, most probably, due to the complex process of interaction of precipitation, melt water, and firn in the surface layer of the glacier.

3. The error in determining the surface temperature can be significant by virtue of the fact that the mathematical model used fails to take into account all possible physical processes in the glacier. The error of the deviation of the experimental profile from a profile corresponding to the reconstructed boundary temperature can reach several percent. But the corresponding error in determining the surface temperature can amount to tens of percent. It should be taken into account that temperature measurements in a well can contain significant errors (as a rule, the instrument accuracy of measuring the temperature is very high, but the measured values of the temperatures can differ from real ones owing to disturbances caused in the process of drilling of the well).

4. The initial temperature profile in the past is one parameter of the inverse problem. The initial temperature distribution actually has no effect on the reconstructed temperature of the surface after the time t_0, which depends on the glacier height. For glaciers with a height of approximately 500 m, this time is a magnitude of the order of 5000 years.

5. The differences in the reconstructed boundary temperatures for the glaciers under study in the past 200 years are insignificant (the accumulation rate lies in the interval from 0 to 0.6 m/year, while the interval in which the value of the geothermal heat flux can lie has boundaries from 0.03 to 0.06 W/m^2).

6. The temperature distribution near the base in the Akademiya Nauk, Franz Josef Land, Barnes Ice-cap, and Gulia Glaciers coincides with the stationary distribution of the temperature, and the same distribution in the Austfonna Glacier indicates the complex nature of the interaction of different temperature regimes that had an effect on the formation of the profile in this region. A phase transition, probably, occurs at the glacier–rock interface. According to our evaluations, the melting rate of ice at the interface for the Austfonna Glacier lies in the interval 2–7 mm/year.

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NOTATION

$\mu(t)$, temperature of the glacier surface (temperature at a depth of 10 m); $T(z, t_f)$, profile of the temperature obtained from solution of the primal problem for a heat-conduction equation; R, operator that maps a certain set of functions representing the boundary temperature onto the corresponding set of profiles; H, depth (height) of the glacier; k thermal-conductivity coefficient; χ, thermal diffusivity; t, time; t_f, final instant of time; t_0, dissipation time of the data on the climate in the past; Q, geothermal heat flux; $w(z)$, vertical advection velocity (advection profile); Ψ, smoothing functional; Ω, stabilizing functionals; m_0, order of the stabilizer; $A_n(z)$, amplitude of harmonic oscillations of the surface temperature that penetrate into the glacier; λ, length of the temperature wave; ω_n, frequency of harmonic oscillations of the surface; b_n, coefficients of the trigonometric polynomial (Fourier series) in the form of which the boundary temperature can be represented.

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